

# Infrared divergences

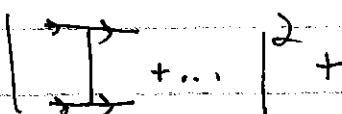
Let's study the  $m \rightarrow 0$  limit. First note that we can set  $\ln(-\frac{t}{m^2}) = \ln(\frac{s}{m^2}) + \ln(-\frac{t}{s}) \approx \ln(\frac{s}{m^2})$  in the limit  $m^2 \ll s, t, u$

Using this trick, we can write

$$T_{m \rightarrow 0} = T_0 \left\{ 1 - \frac{g^2}{128\pi^3} \left[ \frac{11}{6} \ln\left(\frac{s}{m^2}\right) \right] \right\}$$

$\uparrow$  tree-level result

One of the problems is very easy to state and understand. If final-state particles are very soft or collinear to another particle, they are experimentally indistinguishable. Physically, what is measurable is the sum of the cross sections for  $2 \rightarrow 2$  and  $2 \rightarrow 3$  where the 3rd is soft/collinear.

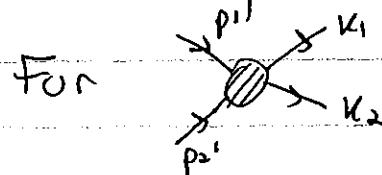
In pictures, must add  +  + ...

Let's begin by writing down the cross section. For the  $2 \rightarrow 2$  scattering we just computed

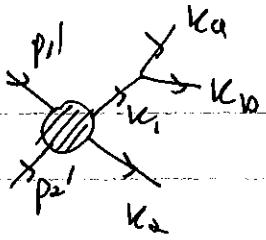
$$d\sigma^{(2 \rightarrow 2)} = \frac{1}{4E_1'E_2'|\vec{v}_1' - \vec{v}_2'|} |T_{m \rightarrow 0}|^2 \frac{d^{d-1}K_1 d^{d-1}K_2}{2E_1' 2E_2' [(2\pi)^{d-1}]^2} (2\pi)^d \delta^{(d)}(\vec{p}_1' \vec{p}_2' - \vec{k}_1' \vec{k}_2')$$

For  $m \rightarrow 0$ ,  $|\vec{v}_1' - \vec{v}_2'| \approx 2$ , and  $8E_1'E_2' \approx 8\left(\frac{\sqrt{s}}{2}\right)\left(\frac{\sqrt{s}}{2}\right) = 2s$

$$\Rightarrow d\sigma^{(2 \rightarrow 2)} = \frac{1}{2s} |T_{m \rightarrow 0}|^2 \frac{d^{d-1}K_1 d^{d-1}K_2}{4E_1'E_2' [(2\pi)^{d-1}]^2} (2\pi)^d \delta^{(d)}(\vec{p}_1' + \vec{p}_2' - \vec{k}_1' - \vec{k}_2')$$



Now add on the process



The amplitude for this is related to that of the 2 $\rightarrow$ 2 process

$$T^{(2 \rightarrow 3)} = T^{(2 \rightarrow 2)} i g \frac{c}{k_1^2 - m^2}.$$

For simplicity denote the L.I.P.S of each particle as

$d^{d-1} k = \hat{d} k$

$2E_k (\hat{d} \pi)^{d-1}$

+ other such splittings

$$\text{Then, we can write } d\sigma^{(2 \rightarrow 2)} = \frac{1}{2s} |T^{(2 \rightarrow 2)}|^2 \hat{d} k_1 \hat{d} k_2 (2\pi)^d \delta^{(d)}(\vec{p}_1 + \vec{p}_2 - \vec{k}_1 - \vec{k}_2)$$

$$d\sigma^{(2 \rightarrow 3)} = \frac{1}{2s} |T^{(2 \rightarrow 2)}|^2 \left\{ \frac{g^2}{(k_1^2 - m^2)^2} \right\} \hat{d} k_2 \frac{1}{2} \hat{d} k_a \hat{d} k_b (2\pi)^d \delta^{(d)}(\vec{p}_1 + \vec{p}_2 - \vec{k}_a - \vec{k}_b - \vec{k}_2)$$

+ ... identical particles that we will integrate over in  $\frac{1}{2}$  account for  $k_a < k_b$

$$\text{Insert unity in the form } \hat{d} k_1 2E_1 (\hat{d} \pi)^{d-1} \delta^{(d-1)}(\vec{k}_1 - \vec{k}_a - \vec{k}_b) = 1$$

$$\Rightarrow d\sigma^{(2 \rightarrow 3)} = \frac{1}{2s} |T^{(2 \rightarrow 2)}|^2 \left\{ \frac{g^2}{(k_1^2 - m^2)^2} \right\} \hat{d} k_1 \hat{d} k_2 \frac{1}{2} \hat{d} k_a \hat{d} k_b$$

$$2E_1 (\hat{d} \pi)^{d-1} \delta^{(d-1)}(\vec{k}_1 - \vec{k}_a - \vec{k}_b) (2\pi)^d \delta^{(d)}(\vec{p}_1 + \vec{p}_2 - \vec{k}_1 - \vec{k}_2)$$

If we now combine these into an observable cross section,

$$\text{we have } d\sigma^{(\text{obs})} = d\sigma^{(2 \rightarrow 2)} + d\sigma^{(2 \rightarrow 3)}$$

$$= \frac{1}{2s} |T^{(2 \rightarrow 2)}|^2 \hat{d} k_1 \hat{d} k_2 (2\pi)^d \delta^{(d)}(\vec{p}_1 + \vec{p}_2 - \vec{k}_1 - \vec{k}_2)$$

$$\left\{ 1 + \frac{g^2}{(k_1^2 - m^2)^2} (2\pi)^{d-1} 2E_1 \delta^{(d-1)}(\vec{k}_1 - \vec{k}_a - \vec{k}_b) \frac{1}{2} \hat{d} k_a \hat{d} k_b \right\}$$

Study the integral over  $k_a, k_b$ .  $k_i^2 = (k_a + k_b)^2$

$$= 2m^2 + 2k_a \cdot k_b = 2m^2 + 2E_a E_b$$

$$\{1 - 2E_a E_b \cos\theta\}$$

In the massless limit,  $k_i^2 \approx 2E_a E_b (1 - \cos\theta)$

Study  $\frac{d^5 k_a d^5 k_b}{(k_i^2 - m^2)^2} \sim \frac{dE_a E_a^{d-3} dE_b E_b^{d-3} d(\cos\theta) (\sin\theta)^{d-4}}{E_a^2 E_b^2 (1 - \cos\theta)^2}$

Soft limits:  $dE_a E_a^{d-5} \rightarrow$  diverges as  $E_a \rightarrow 0$  For  $d \leq 4$

Collinear limit:  $\frac{d(\cos\theta) (\sin\theta)^{d-4}}{(1 - \cos\theta)^2} \sim d\theta \theta^{d-7} \rightarrow$  diverges as  $\theta \rightarrow 0$  for  $d \leq 6$

Currently, we only have to deal with the collinear singularity  
Do so by integrating over a small cone  $0 \leq \beta \leq \delta \Rightarrow \delta$   
depends on properties of detectors. We'll see that the mass cuts off this singularity.

$$(2\pi)^{d-1} 2E_i \delta^{(d-1)}(\vec{k}_i - \vec{k}_a - \vec{k}_b) \frac{1}{2} \frac{d^5 k_a}{2E_a (2\pi)^d} \frac{d^5 k_b}{2E_b (2\pi)^d}$$

$$= \frac{1}{4(2\pi)^5} \frac{E_{\perp}}{E_a E_b} d^5 k_a \quad \text{with } k_b = k - k_a$$

$$\text{Now set } d^5 k_a = |\vec{k}_a|^4 d\vec{k}_a d\theta \sin^3\theta$$

$$k_i = (E_i, \vec{\sigma}, |\vec{k}_i|)$$

$$k_a = (E_a, \vec{\sigma}, k_a \sin\theta, k_a \cos\theta)$$

$\int_0^{\pi} d\theta$   
4-d solid angle  
From other angular integrations

Define angle w.r.t.  $\vec{k}_i$  direction

$$\Rightarrow d\sigma^{(\text{obs})} = \frac{1}{ds} |T^{(2\rightarrow 2)}|^2 d\vec{k}_1 d\vec{k}_2 (2\pi)^d \delta^{(d)}(p_1' + p_2' - k_1 - k_2)$$

$$\left\{ 1 + \frac{g^2 \gamma_4}{4(2\pi)^5} \int d|\vec{k}_{\text{rel}}| |\vec{k}_{\text{rel}}|^4 d\theta \sin^3 \theta \frac{E_1}{E_a E_b} \frac{1}{(k_1^2 - m^2)^2} \right\}$$

Focus on integral. Note  $|\vec{k}_{\text{rel}}| = |\vec{k}_1 - \vec{k}_2|$ , so  $0 \leq |\vec{k}_{\text{rel}}| \leq |\vec{k}_1|$

Set  $|\vec{k}_{\text{rel}}| = x |\vec{k}_1|$ , so that  $\int_0^{|\vec{k}_1|} d|\vec{k}_{\text{rel}}| \rightarrow |\vec{k}_1| \int_0^1 dx$

Expand denominator to leading non-singular order in  $m, \theta$

$$k_1^2 - m^2 = m^2 + 2E_a E_b - 2\vec{k}_a \cdot \vec{k}_b \Rightarrow \text{need some geometry to get angle between } \vec{k}_a, \vec{k}_b$$

$$\frac{\sin \beta}{|\vec{k}_1|} = \frac{\sin \theta}{|\vec{k}_b|}$$

$$\Rightarrow \beta \approx \frac{\theta}{F_x} \text{ For small } \theta$$

$$k_1^2 - m^2 = m^2 + 2E_a E_b - 2|\vec{k}_a||\vec{k}_b| \cos \beta$$

$$1 - \frac{\beta^2}{2} \approx 1 - \frac{\theta^2}{2(F_x)^2}$$

$$\approx m^2 \left[ \frac{1+x+x^3}{x(F_x)} \right] + |\vec{k}_1|^2 \frac{x}{F_x} \theta^2$$

Do the integral over  $\theta$ :

$$\int_0^{\delta(F_x)} \frac{\theta^3}{\left[ m^2 \left( \frac{1+x+x^3}{x(F_x)} \right) + |\vec{k}_1|^2 \frac{x}{F_x} \theta^2 \right]^2} d\theta$$

IF  $\beta \leq \delta$ ,  $\theta \leq \delta(F_x)$

$$= \frac{(1-x)^2}{2x^2 |\vec{k}_1|^4} \ln \left\{ 1 + \frac{|\vec{k}_1|^2}{m^2} \frac{x^2 (1-x)^2}{1-x+x} \delta^2 \right\}$$

$$= \frac{(1-x)^3}{2|\vec{k}_1|^2 x} \frac{\delta^2}{\left[ m^2 \left( \frac{1-x+x^2}{x(1-x)} \right) + x(1-x) |\vec{k}_1|^2 \delta^2 \right]} \quad \text{Take } m \rightarrow 0$$

$$= \frac{(1-x)^2}{2x^2 |\vec{k}_1|^4} \left\{ \ln \left[ \frac{|\vec{k}_1|^3}{m^2} \frac{x^2 (1-x)^2}{1-x+x^2} \delta^2 \right] - \frac{1}{2} \right\}$$

Go back to  $d\sigma^{(obs)}$  → can set  $m=0$ , as the limit is non-singular <sup>in other terms</sup>

$$\vec{k}_1 \left( \int_0^1 dx \frac{1}{|\vec{k}_1|^4} \right) \stackrel{E_a E_b}{\rightarrow} |\vec{k}_1|^4 \int_0^1 dx \frac{x^3}{1-x} d\sigma^{(2 \rightarrow 2)}$$

$$\Rightarrow d\sigma^{(obs)} = \frac{1}{2\theta} |\Gamma^{(2 \rightarrow 2)}|^2 \widehat{d\vec{k}_1} \widehat{d\vec{k}_2} (2\pi)^d \delta^{(d)}(p_1' + p_2' - k_1 - k_2)$$

$$\left\{ 1 + \frac{g^2 \mathcal{D}_4}{4(2\pi)^5} \frac{1}{2} \int_0^1 dx x(1-x) \left[ 2 \left[ \frac{|\vec{k}_1|^3}{m^2} \delta^2 \frac{x^2 (1-x)^2}{1-x+x^2} \right] - 1 \right] \right\}$$

Use  $\mathcal{D}_4 = 2\pi^2$ , keep only singular pieces

$$d\sigma^{(obs)} = d\sigma^{(2 \rightarrow 2)} \left\{ 1 + \frac{\theta^2}{128\pi^3} \frac{1}{6} 2 \left[ \frac{\delta^2 |\vec{k}_1|^3}{m^2} \right] + C \right\} \quad \text{Tr constant}$$

In the massless limit,  $|\vec{k}_1| = \frac{\sqrt{s}}{2}$ .

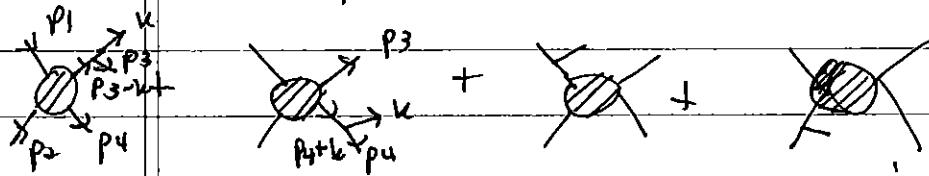
Separate out explicitly the  $\ln(\frac{s}{m^2})$  pieces in  $d\sigma^{(2 \rightarrow 2)}$   
 → multiply this correction by 4 for the 4 external legs \* → show later

$$d\sigma^{(\text{obs})} = d\sigma^{(0)} \left\{ 1 - \frac{\alpha^2}{128\pi^3} \frac{11}{3} \ln\left(\frac{s}{m^2}\right) \right\} \left\{ 1 + \frac{\alpha^3}{128\pi^3} \frac{2}{3} \ln\left[\frac{\delta s}{m^2}\right] \right\}$$

$$= d\sigma^{(0)} \left\{ 1 - \frac{\alpha^2}{128\pi^3} \left[ 3 \ln\left(\frac{s}{m^2}\right) + \frac{2}{3} \ln\left(\frac{1}{\delta^2}\right) \right] \right\}$$

$\Rightarrow \ln\left(\frac{s}{m^2}\right)$  doesn't cancel...

Consider all possible emissions:



$$\Rightarrow T = T^{(2 \rightarrow 2)} \text{ ig} \left\{ \frac{i}{(p_3+k)^2 - m^2} + \frac{i}{(p_4+k)^2 - m^2} + \frac{i}{(p-k)^2 - m^2} \right. \\ \left. + \frac{i}{(p_2-k)^2 - m^2} \right\}$$

$$\text{In massless limit, } p_1 = \frac{\sqrt{s}}{2} (1, \vec{0}, 1) \quad p_3 = \frac{\sqrt{s}}{2} (1, \vec{n}_3) \\ p_2 = \frac{\sqrt{s}}{2} (1, \vec{0}, -1) \quad p_4 = \frac{\sqrt{s}}{2} (1, \vec{n}_4) \\ k = E (1, \vec{n}_k)$$

First note that interferences don't cause divergences

$$dk = E^{d-3} d\cos\theta (\sin\theta)^{d-4}$$

$$\text{For } \frac{1}{(p+k)^2 - m^2} \frac{1}{(p_2-k)^2 - m^2} \equiv \frac{1}{4p_1 k p_2 k}, \text{ let } \theta \text{ be measured} \\ \text{From } p_1$$

$$\alpha_{pi} h \approx E \sqrt{s} (1 - \cos\theta) \quad \alpha_{pik} h \approx E \sqrt{s} (1 + \cos\theta)$$

$$\frac{dk}{4\pi h p_{ik}} \approx \frac{E^{d-5}}{s} \frac{\sin\theta (\sin\theta)^{d-4}}{(1 - \cos\theta)(1 + \cos\theta)}$$

Near  $\theta=0$ :  $\frac{d\theta}{\theta \cdot d} \Rightarrow$  Finite for  $d=6$

Similarly,  $\theta \approx \pi$  is Finite

Similar result for other emission interferences. Can show that initial-state emissions give the same factor.

MS renormalization

To see the second problem leading to the mass divergence, go back & consider our propagator before renormalizing.

$$\Pi(p^2) = -(2\phi-1)p^2 + (2m-1)m^2 + \frac{g^2}{128\pi^3} \left\{ \left( \frac{2}{\varepsilon} + 1 \right) [m^2 - \frac{p^2}{6}] - \int_0^1 dx [m^2 - x(1-x)p^2] \ln \left[ \frac{m^2 - x(1-x)p^2}{u^2} \right] \right\}$$

$$\Pi'(p^2) = -(2\phi-1) + \frac{g^2}{128\pi^3} \left\{ -\frac{1}{6} \left( \frac{2}{\varepsilon} + 1 \right) + \int_0^1 dx x(1-x) \left[ \ln \frac{m^2 - x(1-x)p^2}{u^2} \right] \right\}$$

$$\text{For } m=0, \quad \Pi(p^2) = -(2\phi-1)p^2 + \frac{g^2}{128\pi^3} \left\{ -\frac{p^2}{6} \left( \frac{2}{\varepsilon} + 1 \right) + p^2 \int_0^1 dx x(1-x) \ln \left[ \frac{-x(1-x)p^2}{u^2} \right] \right\}$$

$$\Pi'(p^2) = -(2\phi-1) + \frac{g^2}{128\pi^3} \left\{ -\frac{1}{6} \left( \frac{2}{\varepsilon} + 1 \right) + \int_0^1 dx x(1-x) \ln \left[ \frac{-x(1-x)p^2}{u^2} \right] \right\}$$

Note that  $\Pi(m^2) \rightarrow \Pi(0) = 0$  automatically, without need for renormalization. However,  $\Pi'(0)$  diverges like  $\ln(0)$   $\Rightarrow$  no way to impose  $\Pi'(0) = 0$ . We must switch our renormalization scheme. A useful one is modified minimal subtraction. It is defined by choosing  $2\phi, 2m, 2g$  to remove only the divergent pieces.

$$\text{For example, here we would choose } Z_m = 1 - \frac{g^2}{64\pi^3} \frac{1}{\epsilon}$$

$$Z_\phi = 1 - \frac{1}{8} \frac{g^2}{64\pi^3} \frac{1}{\epsilon}$$

$$\Pi_{\overline{mS}}(p^2) = \frac{g^2}{128\pi^3} \left\{ m^2 - \frac{p^2}{6} - \int_0^1 dx [m^2 - x(1-x)p^2] \ln \left[ \frac{m^2 - x(1-x)p^2}{u^2} \right] \right\}$$

This choice introduces a few big changes that we have to deal with.

$$(1) \text{ The propagator is now } \overline{\Delta}_{\overline{mS}}(p^2) = \frac{i}{p^2 - m^2 - \Pi_{\overline{mS}}(p^2)}$$

The mass of the particle is where  $\overline{\Delta}_{\overline{mS}}(m^2) = 0$

$\Rightarrow$  this is where the K.G. eq.  $(\partial^2 + m_{pn}^2) \overline{\Delta}(x) = 0$  is satisfied. Now, since  $\Pi_{\overline{mS}}(m_{pn}^2) \neq 0$ , the mass is defined by  $m_{pn}^2 - m^2 - \Pi_{\overline{mS}}(m_{pn}^2) = 0 \Rightarrow$  the physical mass  $m_{ph}$  is not the same as the parameter  $m^2$  appearing in the Lagrangian. Note that

$\Pi_{\overline{mS}} = O(g^2)$ , so we can solve for  $m_{pn}^2$  in terms of

$$m^2 \Rightarrow m_{pn}^2 = m^2 \left\{ 1 + \frac{g^2}{128\pi^3} \left[ \frac{5}{6} - \int_0^1 dx (1-x(1-x)) \ln \left( \frac{(1-x(1-x))m^2}{u^2} \right) \right] \right\}$$

(2) This leads us to the 2nd issue:  $m_{ph}$  appears to depend on the parameter  $u \Rightarrow$  can't happen, since it is a physical mass

(3) We also need to note that single-particle states aren't properly normalized anymore

Let's go back and study what we needed for LSZ.

$$\langle 0 | \phi(0) | p \rangle = i \int d^4x e^{-ip \cdot x} (\partial^2 + m_{pn}^2) \langle 0 | T\{\phi(x)\phi(0)\} | 0 \rangle$$

because  $|p\rangle$  is an on-shell state

$$= -i(p^2 - m_{pn}^2) \int d^4x e^{-ip \cdot x} \langle 0 | T\{\phi(x)\phi(0)\} | 0 \rangle$$

must equal unity

Can write the time-ordered product by summing 2-point diagrams

$$\Rightarrow \langle 0 | \phi(0) | p \rangle = (p^2 - m_{pn}^2) \int \frac{d^4k}{(2\pi)^4} \frac{e^{i(k-p) \cdot x}}{k^2 - m^2 - \Pi_{ms}(k^2)}$$

$\int d^4k$

Now expand around  $k^2 = m_{pn}^2$ : First term in denom,

$$m_{pn}^2 - m^2 - \Pi_{ms}(m_{pn}^2) = 0 \text{ by definition, Next term:}$$

$$[m_{pn}^2 - m^2] [1 - \Pi'_{ms}(m_{pn}^2)]$$

$$\Rightarrow \langle 0 | \phi(0) | p \rangle = \frac{1}{1 - \Pi'_{ms}(m_{pn}^2)} \equiv R$$

$$\begin{aligned} \text{denom is } & p^2 - m^2 - \Pi_{ms}(p^2) \\ & = p^2 - m^2 - \Pi_{ms}(m_{pn}^2) \end{aligned}$$

$$\text{residue } = (m_{pn}^2 - m^2)[1 - \Pi']$$

How do we deal with this? Inside the integrand, associate  $R^{-1/2}$  with every field operator

$$\langle 0 | T\{\phi(x)\phi(0)\} | 0 \rangle \Rightarrow \langle 0 | T\{R^{-1/2}\phi(x)R^{-1/2}\phi(0)\} | 0 \rangle$$

$\Rightarrow R^{-1/2}\phi(x)$  creates a properly normalized 1 particle state  
 $\{ |p\rangle \sim \phi(x)|0\rangle \}$

Go back to LSZ reduction, say for  $2 \rightarrow 2$  scattering

$$\langle F | i \rangle = (+i)^4 (d^4 x_1 d^4 x_2 d^4 x'_1 d^4 x'_2) \cancel{e^{ik'_1 \cdot x_1} e^{-ik'_2 \cdot x_2}} e^{-i k'_1 \cdot x'_1} e^{-i k'_2 \cdot x'_2}$$

$$e^{ik'_1 \cdot x'_1} e^{ik'_2 \cdot x'_2} (\partial_1^2 + m p_n^2) (\partial_2^2 + m p_n^2) (\partial_3^2 + m p_n^2)$$

$$(\partial_4^2 + m p_n^2) \langle 0 | T\{\phi(x_1) R^{-1/2} \phi(x_2) R^{-1/2} \phi(x'_1) R^{-1/2}$$

$$\phi(x'_2) R^{-1/2}\} | 0 \rangle$$

From rescaling  
just discussed

At some point in the derivation of the scattering amplitude, we also have  $\left[ \frac{(p^2 - m p_n^2)}{p^2 - m^2 - i \Gamma \bar{m}_S(m^2)} \right]_{p^2 = m p_n^2}$

For each external particle  $\Rightarrow$  gives  $R$ . Combine to get  $\boxed{R^{1/2}}$  For each external state  $\Rightarrow$  this is the rule

$$R = \frac{1}{1 + i \Gamma \bar{m}_S(m^2)} = 1 + i \Gamma \bar{m}_S(m^2) + \dots$$

$$= 1 + \frac{g^2}{128 \pi^3} \frac{1}{6} \ln\left(\frac{m^2}{\mu^2}\right)$$

$$R^{1/2} = 1 + \frac{g^2}{128 \pi^3} \frac{1}{12} \ln\left(\frac{m^2}{\mu^2}\right) \Rightarrow \text{this for each particle in amplitude}$$

There is a  $\ln(m^2)$  in this... maybe it leads to the cancellation of the similar term we found in  $\sigma^{(0hs)}$ ?

One more thing we should do, & that is to renormalize the 3-point vertex in  $\overline{MS}$ .

$$\Rightarrow V_3, \overline{MS} (p_1^2, p_2^2, p_3^2) = g - \frac{g^3}{64\pi^3} \int_0^1 dx_1 dx_2 dx_3 \delta(1-x_1-x_2-x_3) \ln \left[ \frac{m^2 - x_2 x_3 p_3^2 - x_1 x_3 p_1^2 - x_1 x_2 p_2^2}{u^2} \right]$$

We'll go back now & put everything into  $d\sigma^{(\text{obs})}$ . In the virtual corrections,  $iV_3(p_1^2, p_2^2, s) \equiv ig \left\{ 1 - \frac{g^3}{128\pi^3} \ln \left[ \frac{-s}{u^2} \right] \right\}$

now + constant

$$\text{Also, } i\bar{\Delta}(s) \equiv \frac{i}{s} \left\{ 1 + \frac{g^3}{128\pi^3} \frac{1}{6} \ln \left( \frac{-s}{u^2} \right) + c' \right\}$$

in the high energy limit

$$T^{(2 \rightarrow 2)} = T_0 \left\{ 1 - \frac{g^2}{128\pi^3} \frac{11}{6} \ln \left( \frac{s}{u^2} \right) + \text{constant} \right\}$$

$$d\sigma^{(\text{obs})} = d\sigma^{(0)} \left\{ 1 - \frac{2g^2}{128\pi^3} \frac{11}{6} \ln \left( \frac{s}{u^2} \right) \right\} \left\{ 1 + \frac{g^2}{128\pi^3} \frac{2}{3} \ln \left[ \frac{\delta s}{m^2} \right] \right\}$$

$\star R^4$       ↑       $T$   
new  $\overline{MS}$       virtual       $2 \rightarrow 3$  process  
new factor      ↓      From LSZ

$$\text{Set } R^2 = 1 + \frac{g^2}{128\pi^3} \frac{2}{3} \ln \left( \frac{m^2}{u^2} \right)$$

$$\Rightarrow d\sigma^{(\text{obs})} = d\sigma^{(0)} \left\{ 1 + \frac{g^2}{128\pi^3} \left[ -\frac{11}{3} \ln \left( \frac{s}{u^2} \right) + \frac{2}{3} \ln \left( \frac{m^2}{u^2} \right) \right. \right.$$

$$\left. + \frac{2}{3} \ln \delta^2 + \frac{2}{3} \ln \left( \frac{s}{u^2} \right) - \frac{2}{3} \ln \left( \frac{m^2}{u^2} \right) \right] + C'' \left. \right\}$$

$$\Rightarrow \ln\left(\frac{m^2}{\mu^2}\right) \text{ cancels! } d\sigma^{(0\text{bs})} = d\sigma^{(0)} \left\{ 1 + \frac{g^2}{128\pi^3} \left[ -3\ln\left(\frac{\mu}{\mu_0}\right) + \frac{2}{3}\ln(\delta^2) + \mathcal{O}(m^0) \right] + \mathcal{O}(g^4) \right\}$$

Still have dependence on  $u$ , which must cancel

$$d\sigma^{(0)} = \# g^4 \Rightarrow g^2 = g^2(u)$$

$$\frac{d\sigma^{(0\text{bs})}}{d\ln u} = \frac{d\sigma^{(0)}}{d\ln u} + \frac{6g^2(u)d\sigma^{(0)}}{128\pi^3} = 0$$

$$\text{Set } \frac{d\sigma^{(0)}}{d\ln u} = 2g^2(u) \frac{dg^2(u)}{d\ln u} \frac{d\sigma^{(0)}}{g^4(u)}$$

$$\Rightarrow \frac{2}{g^2(u)} \frac{dg^2(u)}{d\ln u} + \frac{6g^2(u)}{128\pi^3} = 0$$

$$\Rightarrow \frac{dg^2(u)}{d\ln u} = -\frac{3}{128\pi^3} g^4(u)$$

we can solve this given an initial condition at some scale  $u_0$ .  $\Rightarrow \frac{dg^2(u)}{g^4(u)} = -\frac{3}{128\pi^3} d\ln(u)$

$$\Rightarrow \frac{1}{g^2(u_0)} - \frac{1}{g^2(u)} = -\frac{3}{128\pi^3} \ln\left(\frac{u}{u_0}\right)$$

$$\Rightarrow g^2(u) = \frac{g^2(u_0)}{1 + \frac{3}{128\pi^3} g^2(u_0) \ln\left(\frac{u}{u_0}\right)}$$

The way to think about this is the following. The process we are considering has the natural choice  $\mu^2 \approx s \Rightarrow$  this minimizes the size of the  $\ln(\frac{s}{\mu^2})$  term, makes the tree-level approximation better. The effect instead appears in the running coupling  $g^2(u)$ , which is a function of energy.

As  $u$  increases,  $g^2(u) \rightarrow 0 \Rightarrow$  weak scattering at high energies. As  $u$  decreases, coupling blows up  $\Rightarrow$  can't use perturbation theory any more. Such a theory is called asymptotically free  $\Rightarrow$  QCD is an example.

If instead  $\frac{dg^2(u)}{d\ln u} > 0$ , then the coupling becomes large at high energies, small at low energies  $\Rightarrow$  QED is an example

## Renormalization group

We can pursue this idea of independence upon changing  $\mu$  more systematically. Write down our original Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi_0 \partial_\mu \phi_0 - \frac{1}{2} m_0^2 \phi_0^2 + \frac{1}{6} g_0 \phi_0^3 + Y_0 \phi_0$$

This is divergent, as we already know, because there are no counterterms  $\Rightarrow$  the Field parameters are called bare fields, parameters. Introduce the counterterms

$$\mathcal{L} = \frac{1}{2} Z\phi \partial^\mu (\bar{\phi})^\mu \phi - \frac{1}{2} Zm m^2 \phi^2 + \frac{1}{6} Zg g \bar{\mu}^{\varepsilon/2} \phi^3 + Y\phi$$

These have the renormalized Field parameters. Simple to see the relationships

$$\begin{aligned} \phi_0 &= Z_\phi^{1/2} \phi & g_0 &= Zg Z_\phi^{-1/2} g \bar{\mu}^{\varepsilon/2} \\ m_0 &= Z_m^{1/2} Z_\phi^{-1/2} m & Y_0 &= Z_\phi^{-1/2} Y \end{aligned}$$

In the  $\overline{\text{MS}}$  renormalization scheme, the renormalization constants depend on  $g, \varepsilon, \epsilon$  nothing else

$$Z_\phi = 1 + \sum_{n=1}^{\infty} \frac{a_n(g)}{\varepsilon^n}, \quad Z_m = 1 + \sum_{n=1}^{\infty} \frac{b_n(g)}{\varepsilon^n},$$

$$Z_g = 1 + \sum_{n=1}^{\infty} \frac{c_n(g)}{\varepsilon^n}$$

We seen the leading terms by explicit calculation

$$a_1 = -\frac{g^2}{64\pi^3} \frac{1}{6} ; b_1 = -\frac{g^2}{64\pi^3} ; c_1 = -\frac{g^2}{64\pi^3}$$

We saw in the last section of notes that independence of the cross section from  $\mu$  led to equations predicting the running of  $g^2(\mu)$ ,  $m(\mu)$ . The trick for calculating these is to use that the bare fields, parameters are independent of  $\mu \Rightarrow$  no such parameter exists in the original Lagrangian, it only appeared at intermediate stages for calculational convenience.

Let's see how to use this. set  $g_0^2 = Zg^2 Z_\phi^{-3} \tilde{\mu}^\varepsilon g^2$

$$\rightarrow \ln(g_0^2) = \ln[Zg^2 Z_\phi^{-3}] + \varepsilon \ln \tilde{\mu} + \ln g^2$$

$$\frac{d}{d \ln \mu} \ln(g_0^2) = \varepsilon + \frac{d}{d \ln \mu} \ln(g^2) + \frac{d}{d \ln \mu} \ln[Zg^2 Z_\phi^{-3}] = 0$$

Write  $G(g^2, \varepsilon) = \ln[Zg^2 Z_\phi^{-3}] = \sum_{n=1}^{\infty} \frac{G_n(g^2)}{\varepsilon^n}$

$$\frac{d G}{d \ln \mu} = \frac{\partial G}{\partial g^2} \frac{d g^2}{d \ln \mu} \Rightarrow \text{plug into above to Find}$$

$$\varepsilon + \frac{\partial G}{\partial g^2} \frac{d g^2}{d \ln \mu} + \underbrace{\frac{d}{d \ln \mu} \ln(g^2)}_{\frac{1}{g^2} \frac{d g^2}{d \ln \mu}} = 0$$

Solve to Find  $\left\{ 1 + \frac{\varepsilon}{g^2} \frac{\partial G}{\partial g^2} \right\} \frac{dg^3}{d\ln u} = -\varepsilon g^2$

$$\Rightarrow \frac{dg^3}{d\ln u} = \frac{-\varepsilon g^2}{1 + \frac{\partial G}{\partial g^2} g^2} \quad \frac{\partial G}{\partial g^2} = \frac{1}{\varepsilon} \frac{\partial G_1}{\partial g^2} + \frac{1}{\varepsilon^2} \frac{\partial G_2}{\partial g^2} + \dots$$

We expect this derivative to be finite; changing the energy scale of the coupling by a small amount shouldn't result in an infinite shift. This indeed turns out to be the case in general. Only the  $G_1$  term matters

$$\text{For this } \Rightarrow \frac{dg^3}{d\ln u} = -\varepsilon g^2 + g^4 \underbrace{\frac{dG_1}{dg^2}}_{B(g^2)}$$

In the  $\varepsilon=0$  limit, this should match what we had before.

$$G = \ln[zg^2 z_\phi^{-3}] = \ln \left\{ \left[ 1 - \frac{2g^3}{64\pi^3 \varepsilon} \right] \left[ 1 + \frac{g^2}{64\pi^3} \frac{1}{2\varepsilon} \right] \right\}$$

$$\approx \ln \left\{ 1 - \frac{g^2}{64\pi^3} \frac{3}{2\varepsilon} \right\}$$

$$\approx 1 - \frac{g^2}{64\pi^3} \frac{3}{2\varepsilon} \Rightarrow G_1 = \frac{3}{2} \frac{g^2}{64\pi^3}$$

$$\Rightarrow B = \frac{3}{2} \frac{1}{64\pi^2} g^4$$

$\Rightarrow$  same that we found in the previous section by directly taking  $\frac{d\Gamma^{(\text{obs})}}{d\ln u} = 0$

We can do the same analysis for other parameters of the theory.  $m_0 = Zm^{1/2} Z_\phi^{-1/2} m$

$$\Rightarrow \ln m_0 = \ln M(g, \varepsilon) + \ln m$$

$$\text{with } M(g, \varepsilon) = \ln(Zm^{1/2} Z_\phi^{-1/2})$$

$$= \sum_{n=1}^{\infty} \frac{M_n(g)}{\varepsilon^n}$$

$$\frac{d \ln m_0}{d \ln u} = 0 = \underbrace{\frac{d \ln(m)}{d \ln u}}_{\frac{1}{m} \frac{dm}{d \ln u}} + \underbrace{\frac{\partial M(g, \varepsilon)}{\partial g^2} \frac{\partial g^2}{d \ln u}}_{-\varepsilon g^2 + B(g^2)}$$

$$\Rightarrow \text{solve to get } \frac{1}{m} \frac{dm}{d \ln u} = \{\varepsilon g^2 - B(g^2)\} \sum_{n=1}^{\infty} \frac{1}{\varepsilon^n} \frac{d}{dg^2} M_n(g)$$

Should be Finite, so only the  $n=1$  term can

$$\text{contribute} \Rightarrow \frac{1}{m} \frac{dm}{d \ln u} = g^2 \cdot \frac{d}{dg^2} M_1(g^2) \equiv \gamma_m(g^2)$$

$$\begin{aligned} M(g, \varepsilon) &= \ln \left\{ \left\{ 1 - \frac{g^2}{64\pi^3} \frac{1}{2} \right\} \left\{ 1 + \frac{g^3}{64\pi^3} \frac{1}{12} \right\} \right\} \\ &= \ln \left\{ 1 - \frac{5}{12} \frac{g^2}{64\pi^3} \right\} \approx -\frac{5}{12} \frac{g^2}{64\pi^3} \end{aligned}$$

$$\Rightarrow \frac{1}{m} \frac{dm}{d \ln u} = -\frac{5}{12} \frac{g^2}{64\pi^3}$$

The way we would use this is as discussed in 2.2 scattering  $\Rightarrow$  use this to absorb  $\ln(\frac{\mu}{\mu_0})$  corrections

(III)

appearing in higher-orders into a running coupling at tree-level.

Can do the same for the propagator. the bare  $\overline{MS}$  propagators are

$$\overline{D}_{\text{us}}(p^2) = i \int d^6x e^{-ik \cdot x} \langle 0 | T \Sigma \phi(x) \phi(0) | 0 \rangle$$

$$\overline{D}_0(p^2) = i \int d^6x e^{-ik \cdot x} \langle 0 | T \{ \phi_0(x) \phi_0(0) \} | 0 \rangle$$

$$\Rightarrow \overline{D}_0 = Z_\phi \overline{D}_{\text{us}} \Rightarrow \frac{d \overline{D}_0}{d \ln u} = 0$$

$$\Rightarrow \frac{d}{d \ln u} \ln \overline{D}_0 = \frac{d \ln Z_\phi}{d \ln u} + \frac{d}{d \ln u} \ln \overline{D}_{\text{us}} = 0$$

Propagator is a function of  $\overline{D}_{\text{us}} = \overline{D}_{\text{us}}[g^2(u), m(u), u]$

$$\Rightarrow \frac{d}{d \ln u} \ln \overline{D}_{\text{us}} = \frac{1}{\overline{D}_{\text{us}}} \left\{ \frac{\partial}{\partial \ln u} + \frac{dg^2}{d \ln u} \frac{\partial}{\partial g^2} + \frac{dm}{d \ln u} \frac{\partial}{\partial m} \right\} \overline{D}_{\text{us}}$$

$$\text{Can also write } \frac{d \ln Z_\phi}{d \ln u} = \frac{\partial}{\partial g^2} \ln Z_\phi \frac{dg^2}{d \ln u}$$

This should be Finite, as for the others

$$\begin{aligned} \frac{dg^2}{d \ln u} \frac{\partial}{\partial g^2} \ln Z_\phi &= \left\{ \frac{1}{\epsilon} \frac{\partial}{\partial g^2} a_1 \right\} \{-\epsilon g^2 + B(g^2)\} \\ &= -g^2 \frac{\partial}{\partial g^2} a_1 \end{aligned}$$

$$\text{Define } \gamma_\phi = \frac{1}{2} \frac{d \ln Z_\phi}{d \ln u} = -\frac{g^2}{2} \frac{\partial}{\partial g^2} a_1$$

We have a differential eq. for the propagator

$$\Rightarrow \left\{ \frac{\partial}{\partial \ln a} + B(g^2) \frac{\partial}{\partial g^2} + \gamma_m(g^2) m \frac{\partial}{\partial m} + 2\gamma_\phi(g^2) \right\} \bar{D}_{\bar{m}\bar{s}} = 0$$

$\Rightarrow$  the Callan-Symanzik eq.